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Rational Interpolation with Restricted Poles*

THOMAS BAGBY

Department of Mathematics, Indiana University, Bloomington, Indiana 47401

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1. SUMMARY OF RESULTS

In this paper we discuss interpolation to an analytic function on a compact set in the extended complex plane $\mathbb{P}$, by rational functions, whose poles lie in a disjoint compact set. To expedite the statement of theorems we agree that for the rest of the paper A and B will denote disjoint compact sets in $\mathbb{P}$, and R will denote $\mathbb{P} - (A \cup B)$.

An *interpolation scheme* $\{a_{ni}, b_{ni}\}$ consists of a collection of points a_{ni} and b_{ni} in $\mathbb{P}$ defined for every positive integer n and every positive integer $i \leqslant n$, with the requirement that $a_{ni} \neq b_{nj}$ for $1 \leqslant i, j \leqslant n$. The *associated potential function* $u_{n,t}(z)$ is defined (unless t and z simultaneously lie in the set $\{a_{n1}, ..., a_{nn}\}$ or in the set $\{b_{n1}, ..., b_{nn}\}$) by the formula

$$u_{n,t}(z) = -\frac{1}{n} \sum_{i=1}^{n} \log |[z, a_{ni}, t, b_{ni}]| ,$$

where the brackets denote the cross-ratio. In some discussions the reference point t is fixed, and we simplify the notation by writing $u_n = u_{n,t}$. Let $\{a_{ni}, b_{ni}\}$ be an interpolation scheme, and f an analytic function defined on an open set containing the points $a_{n1}, ..., a_{nn}$; we say that the $(n-1)$-degree rational function r_n interpolates to f via the interpolation scheme provided r_n coincides with f at the points $a_{n1}, ..., a_{nn}$ and has the poles $b_{n1}, ..., b_{n\,n-1}$, with the usual agreements concerning multiplicities. A central problem is to relate the interpolating rational functions and the associated potentials.

An interpolation scheme $\{a_{ni}, b_{ni}\}$ lies *on* A, B if all points a_{ni} are in A and all points b_{ni} are in B. The case $B = \{\infty\}$ corresponds to the problem of polynomial approximation. This problem has been studied by many authors [6, Chapter 7], and the following definitive result is due to Walsh [6, Sections 7.2–7.4].

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THEOREM 1. *Let A be a nonempty compactum in $\mathbb{P}$ such that $R = \mathbb{P} - A$ is connected and regular for the Dirichlet problem and contains $B = \{\infty\}$. Let u denote the unique harmonic function in R which has constant boundary values at A and satisfies* $\lim_{z\to\infty}(u(z) + \log | z |) = 0$; *and let* $V = u(A)$. *Let* $\{a_{ni}, \infty\}$ *be an interpolation scheme on A, B, and $u_n = n_{n,\infty}$ the associated potentials. Then the following statements are equivalent.*

(1a) *As $n \to \infty$ we have* $\inf_A u_n \to V$.

(1b) *As $n \to \infty$ we have $u_n \to u$ uniformly on compacta in R.*

(1c) *If f is any analytic function on A, and P_n interpolates to f via $\{a_{ni}, \infty\}$, then $P_n \to f$ uniformly on A.*

We turn now to the analogous theory for rational approximation. One setting for the theory [5; 6, Chapter 8] involves the "dual" problems of approximating analytic functions on A by rational functions having poles on B, and approximating analytic functions on B by rational functions having poles on A. Using a very general topological setting, Walsh [6, Section 8.3, Corollary 2] showed that a condition analogous to (1b) is sufficient for the dual approximation property, and the present author [1, Theorem 2] showed that Walsh's condition, and a condition analogous to (1a), are both necessary and sufficient.

The purpose of the present paper is to discuss analogous results for the "one-sided" problem of rational interpolation [3; 6, Chapter 9], where we are required to approximate analytic functions on A but not analytic functions on B, using a topological setting of similar generality. We summarize our results in the following two theorems. The first is a very general description of the interrelationship between the behavior of the potentials u_n and the power of the interpolation scheme $\{a_{ni}, b_{ni}\}$ for approximating analytic functions.

THEOREM 2. *Let A and B be nonempty, disjoint compacta in $\mathbb{P}$ such that $\mathbb{P} - A$ is connected and regular for the Dirichlet problem. Let $\{a_{ni}, b_{ni}\}$ be an interpolation scheme on A, B and u_n the associated potentials with respect to a fixed reference point in $R = \mathbb{P} - (A \cup B)$. Then the following statements are equivalent.*

(2a) *If a subsequence u_{n_k} converges uniformly on compacta in R, then the limit function u has a constant boundary value at A.*

(2b) *If f is any analytic function on A, and r_n interpolates to f via $\{a_{ni}, b_{ni}\}$, then $r_n \to f$ uniformly on A.*

We next state a theorem which has a form closer to the forms of Theorem 1 and a theorem given by Shen in a different setting [3; 6, Section 9.9]. For this we need to introduce some more notation. If $\mathbb{P} - A$ is regular for the Dirichlet problem, and if b and t are distinct points in $\mathbb{P} - A$, we let $p_{b,t}$ be the unique harmonic function in $\mathbb{P} - A - \{b\}$ which has a constant boundary value at A, vanishes at the point t, and satisfies

$$\lim_{z\to b} p_{b,t}(z) - \log | z - b | \text{ is finite, if } b \neq \infty,$$

$$\lim_{z\to b} p_{b,t}(z) + \log | z | \text{ is finite, if } b = \infty.$$

We let $V_{b,t}$ denote the constant boundary value of $p_{b,t}$ at A. Now if $\{a_{ni}, b_{ni}\}$ is an interpolation scheme on A, B, then for each n the potential function $p_{n,t} = (1/n) \sum_i p_{b_{ni},t}$ is harmonic throughout all of $\mathbb{P} - A$ except for the points $b_{ni} (i \leqslant n)$ and has the constant boundary value $W_{n,t} = (1/n) \sum_i V_{b_{ni},t}$ at A. In particular, if A and B are bounded, we may think heuristically of $p_{n,\infty}$ as the logarithmic potential due to a unit positive charge on A which has settled into equilibrium under the influence of a fixed unit negative charge equally divided among the points b_{ni} $(i \leqslant n)$. In some discussions the reference point t is fixed, and we simplify the notation by writing $p_n = p_{n,t}$ and $W_n = W_{n,t}$.

THEOREM 3. *Let the hypotheses of Theorem 2 hold, and let the data* u_n, p_n, W_n *be taken with respect to the same fixed reference point in* R. *Then the following statements are equivalent.*

(3a) *As* $n \to \infty$ *we have* $\inf_A u_n - W_n \to 0$.

(3b) *As* $n \to \infty$ *we have* $u_n - p_n \to 0$ *uniformly on compacta in* R.

(3c) *If* f *is any analytic function on* A, *and* r_n *interpolates to* f *via* $\{a_{ni}, b_{ni}\}$, *then* $r_n \to f$ *uniformly on* A.

Theorem 2 gives a general characterization, in terms of the associated potentials u_n, of the interpolation schemes $\{a_{ni}, b_{ni}\}$ which can be used to approximate every analytic function on A. On the other hand, if poles b_{ni} are prescribed, we may use them to compute the data p_n and W_n; then Theorem 3 gives a characterization, in terms of the associated potentials u_n, of the interpolation points a_{ni} which can be used with these poles to approximate every analytic function on A. We will find it convenient to give a unified proof of these theorems in Section 3, following some preliminary results on measure–theoretic potential theory in Section 2.

Under the hypotheses of Theorems 2 and 3, there always exist interpolation schemes $\{a_{ni}, b_{ni}\}$ for which the equivalent statements of the theorems hold true. In fact, if any poles b_{ni} $(i \leqslant n)$ are preassigned, then there exist points a_{ni} $(i \leqslant n)$ such that the equivalent statements of the theorems hold true. This may be proved by methods of Shen [3; 6, Chapter 9] and Walsh

[6, Chapter 9], who studied the rate of convergence of rational functions interpolating to analytic functions. Part of the emphasis of the present paper is in obtaining converse theorems under the weakest possible assumptions, and in this sense our work is complementary to that of Shen and Walsh.

2. Potential Theory in the Presence of a Fixed Charge

In the proofs of Theorems 2 and 3 we will use a generalized form of potential theory which we describe in the present section. We assume that the disjoint compacta A and B are bounded, and that $\mathbb{P} - A$ is regular for the Dirichlet problem.

In this paper the word *measure* will signify a signed Borel measure on $\mathbb{P}$. If $E \subset \mathbb{P}$ is compact we let $\mathscr{M}(E)$ denote the set of all unit positive measures with support in E.

If σ is any measure with bounded support, we introduce the potential function

$$u_\sigma(z) = \int \log \frac{1}{|z - t|} \, d\sigma(t),$$

wherever it is defined. If $\{a_{ni}, b_{ni}\}$ is an interpolation scheme on A, B, we use the notation α_n for the unit positive measure on A with $1/n$ units of mass at a_{ni} $(i \leqslant n)$, and β_n for the unit positive measure on B with $1/n$ units of mass at b_{ni} $(i \leqslant n)$. Then we clearly have $u_{n,\infty} = u_{\alpha_n - \beta_n}$.

The following theorem describes mathematically a charge μ which has settled into equilibrium on A, in the presence of a fixed charge $-\nu$ on B. It is analogous to a well known result of Frostman [2; 4, Chapter III], and may be proved by the same techniques.

Theorem A. *Let $\nu \in \mathscr{M}(B)$. Then there exists a unique measure $\mu \in \mathscr{M}(A)$ for which the reduced energy integral*

$$J(\alpha) = \int u_{\alpha - \nu} \, d\alpha, \qquad \alpha \in \mathscr{M}(A),$$

assumes its minimum value K. Moreover, the potential $u_{\mu-\nu}$ is continuous at all points of $\mathbb{P} - B$ and $u_{\mu-\nu}(z) = K$ for all $z \in A$.

We indicate the dependence of μ and K on ν in this theorem by writing $\mu = \mathscr{L}[\nu]$ and $K = \mathscr{K}[\nu]$. There is a close connection between these concepts and those introduced in the preceding section. For example, if ν is the discrete measure consisting of $1/n$ units of mass at each point β_{ni} $(i \leqslant n)$, then $p_{n,\infty} = u_{\mathscr{L}[\nu]}$ and $W_{n,\infty} = \mathscr{K}[\nu]$.

The following convergence theorem will be needed later.

THEOREM B. *Let $\{\nu_n\}$ be a sequence of measures in $\mathscr{M}(B)$, and suppose that $\nu_n \to \nu$ in the weak-star topology. Then $\mathscr{L}[\nu_n] \to \mathscr{L}[\nu]$ in the weak-star topology and $\mathscr{K}[\nu_n] \to \mathscr{K}[\nu]$.*

Proof. We use the notation $\mu_n = \mathscr{L}[\nu_n]$, $\mu = \mathscr{L}[\nu]$, $K_n = \mathscr{K}[\nu_n]$ and $K = \mathscr{K}[\nu]$. Suppose we have a sequence of indices n for which μ_n converges in the weak-star topology and K_n converges: $\mu_n \to \gamma$ and $K_n \to C$. We first observe that

$$K_n \leqslant \int u_{\mu-\nu_n}\, d\mu,$$

and hence

$$C \leqslant \int u_{\mu-\nu}\, d\mu = K.$$

On the other hand, we have for each $z \in A$ the estimate

$$\begin{aligned} C = \lim u_{\mu_n-\nu_n}(z) &= \liminf \int \log \frac{1}{|z-t|}\, d(\mu_n - \nu_n)(t) \\ &\geqslant \int \log \frac{1}{|z-t|}\, d(\gamma - \nu)(t) \\ &= u_{\gamma-\nu}(z), \end{aligned}$$

and hence $K \leqslant \int u_{\gamma-\nu}\, d\gamma \leqslant C$. Thus $K = C$, and since $\int u_{\gamma-\nu}\, d\gamma = K$ we have $\gamma = \mu$. This proves Theorem B.

3. Proof of Theorems 2 and 3

Since these theorems are invariant under Möbius transformations, we may assume that A and B are bounded sets and that the data u_n, p_n, W_n are taken with the reference point at infinity.

(3a) $\Rightarrow$ (3b). If (3b) fails, we can find a sequence of indices n such that $u_n - p_n$ is bounded away from zero at some point of R. We can find a subsequence of these indices, by the theory of normal families, such that $u_n - p_n$ converges to a harmonic function g, uniformly on compact subsets of $\mathbb{P} - A$. If $\epsilon > 0$ is arbitrary, we see from (3a) that $u_n - p_n \geqslant -\epsilon$ in $\mathbb{P} - A$ for n sufficiently large. It follows that $g \geqslant 0$ in $\mathbb{P} - A$. However, $g(\infty) = 0$, since each of the functions $u_n - p_n$ vanishes at ∞, and we conclude that g is identically zero in $\mathbb{P} - A$. This is a contradiction.

(3b) $\Rightarrow$ (2a). This follows from Theorem B.

(2a) $\Rightarrow$ (3c). If (3c) fails, we can find a sequence of indices n such that $\sup_A |f - r_n|$ is uniformly bounded away from zero. We can find a subsequence of these indices such that $\alpha_n \to \alpha$ and $\beta_n \to \beta$ in the weak-star topology. From (2a) we see that $u_{\alpha-\beta}(z) \to C$ as $z \in R$ approaches A. Since $u_{\alpha-\beta}$ is subharmonic in $\mathbb{P} - A$, we have $\sup_B u_{\alpha-\beta} = C_1 < C$. Now if $C_1 < C_2 < C_3 < C$ and if $\gamma_j = \{z \in R : u_{\alpha-\beta} = C_j\}$, then we have the following explicit formula [6, Section 8.1, Eq. (4)] for $z \in \gamma_3$:

$$f(z) - r_n(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(t)}{t - z} \frac{\prod_{i \leqslant n}(z - a_{ni})}{\prod_{i \leqslant n}(t - a_{ni})} \frac{\prod_{i \leqslant n-1}(t - b_{ni})}{\prod_{i \leqslant n-1}(z - b_{ni})} \, dt.$$

Thus, if $\epsilon > 0$ is arbitrary, we have for large n

$$\begin{aligned} \sup_A |f - r_n| &\leqslant \sup_{\gamma_3} |f - r_n| \\ &\leqslant \frac{L(\gamma_2)}{2\pi} \frac{\sup_{\gamma_2} |f|}{d(\gamma_2, \gamma_3)} \frac{\operatorname{diam} \gamma_2 \cup B}{d(\gamma_2, B)} \exp(-n(C_3 - C_2 - \epsilon)), \end{aligned}$$

where $L(\gamma_2)$ denotes the length of γ_2. In particular we have $\sup_A |f - r_n| \to 0$, which is a contradiction.

(3c) $\Rightarrow$ (3a). We begin with the observation that for each n the function $u_n - p_n$ is harmonic in $\mathbb{P} - A$ and vanishes at ∞; we conclude that

$$\inf_A u_n \leqslant W_n .$$

If (3a) fails, we can find a sequence of indices n such that $\inf_A u_n - W_n$ is bounded away from zero. We can find a subsequence of these indices for which $\alpha_n \to \alpha$ and $\beta_n \to \beta$ in the weak-star topology, and thus by Theorem B, μ_n converges to $\mu = \mathscr{L}[\beta]$ in the weak-star topology, and W_n converges to $W = \mathscr{K}[\beta]$. Let $\epsilon > 0$ be arbitrary. Since $u_{\alpha-\beta} - u_{\mu-\beta}$ is harmonic in $\mathbb{P} - A$ and has the value 0 at ∞, we can find a point $t \in R - \{\infty\}$ such that $u_{\alpha-\beta}(t) > W - \epsilon$, and hence $u_n(t) > W - \epsilon$ for n sufficiently large. We now apply (3c) to the function $f(z) = 1/(t - z)$. We have the formula [6, Section 8.1, Eq. (3)]

$$f(z) - r_n(z) = \frac{1}{t - z} \frac{\prod_{i \leqslant n}(z - a_{ni})}{\prod_{i \leqslant n}(t - a_{ni})} \frac{\prod_{i \leqslant n-1}(t - b_{ni})}{\prod_{i \leqslant n-1}(z - b_{ni})},$$

and, hence,

$$\lim_n n \inf_{z \in A} [u_n(z) - u_n(t)] = +\infty.$$

It follows that for n sufficiently large,

$$\inf_{z \in A} u_n(z) \geqslant u_n(t) > W - \epsilon.$$

Since ϵ was arbitrary, we have

$$\lim_n \inf_{z \in A} u_n(z) = W,$$

a contradiction.

References

1. T. Bagby, On interpolation by rational functions, *Duke Math. J.* **36** (1969), 95–104.
2. O. Frostman, Potentiel d'équilibre et capacité des ensembles, *Comm. Sem. Math. Lund* **3** (1935), 1–118.
3. Y. Shen, On interpolation and approximation by rational functions with preassigned poles, *J. Chinese Math. Soc.* **1** (1936), 154–173.
4. M. Tsuji, "Potential Theory in Modern Function Theory," Maruzen, Tokyo, 1959.
5. J. L. Walsh, A duality in interpolation to analytic functions by rational functions, *Proc. Nat. Acad. Sci. U.S.A.* **19** (1933), 1049–1053.
6. J. L. Walsh, Interpolation and approximation by rational functions in the complex domain, *Amer. Math. Soc. Coll. Publ.* **20** (1935).